THE SEGAL-BARGMANN TRANSFORM ON COMPACT SYMMETRIC SPACES AND THEIR DIRECT LIMITS

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Abstract. We study the Segal-Bargmann transform, or the heat transform, H_t for a compact symmetric space $M = U/K$. We prove that H_t is a unitary isomorphism $H_t : L^2(M) \to \mathcal{H}_t(M_{\mathbb{C}})$ using representation theory and the restriction principle. We then show that the Segal-Bargmann transform behaves nicely under propagation of symmetric spaces. If $\{M_n = U_n/K_n, \iota_{n,m}\}_n$ is a direct family of compact symmetric spaces such that M_m propagates M_n, $m \geq n$, then this gives rise to direct families of Hilbert spaces $\{L^2(M_n), \gamma_{n,m}\}$ and $\{\mathcal{H}_t(M_{n\mathbb{C}}), \delta_{n,m}\}$ such that $H_{t,m} \circ \gamma_{n,m} = \delta_{n,m} \circ H_{t,n}$. We also consider similar commutative diagrams for the K_n-invariant case. These lead to isometric isomorphisms between the Hilbert spaces $\varinjlim L^2(M_n) \simeq \varinjlim \mathcal{H}(M_{n\mathbb{C}})$ as well as $\varinjlim L^2(M_n)^{K_n} \simeq \varinjlim \mathcal{H}(M_{n\mathbb{C}})^{K_n}$.

Introduction

Denote by $h_t(x) = (4\pi t)^{-n/2} e^{-\|x\|^2/4t}$ the heat kernel on $\mathbb{R}^n$ and denote by $d\mu_t(x) = h_t(x)dx$ the heat kernel measure on $\mathbb{R}^n$. Denote by Δ the Laplace operator on $\mathbb{R}^n$. The Segal-Bargmann transform H_t, also called the heat kernel transform, on $L^2(\mathbb{R}^n)$ or on $L^2(\mathbb{R}^n, \mu_t)$ is defined by mapping a function $f \in L^2(\mathbb{R}^n)$ to the holomorphic extension to $\mathbb{C}^n$ of $f * h_t = e^{t\Delta} f$. The image of $L^2(\mathbb{R}^n, \mu_t)$ under the Segal-Bargmann transform is the Fock space $\mathcal{F}_t(\mathbb{C}^n)$ of holomorphic functions $F : \mathbb{C}^n \to \mathbb{C}$ such that $(2\pi t)^{-n} \int |F(x + iy)|^2 e^{-\|x+iy\|^2/2t} \, dxdy < \infty$. Thus $\mathcal{F}_t(\mathbb{C}^n) = L^2(\mathbb{R}^{2n}, d\mu_{t/2}(x) d\mu_{t/2}(y)) \cap \mathcal{O}(\mathbb{C}^n)$ whereas the image of $L^2(\mathbb{R}^n)$ is $\mathcal{H}_t(\mathbb{C}^n) = L^2(\mathbb{R}^{2n}, d\mu_{t/2}(y)) \cap \mathcal{O}(\mathbb{C}^n)$, also called the Fock space. This idea, in a slightly different form, was first introduced by V. Bargmann in [3]. An infinite dimensional version was considered by I. E. Segal in [34]. A short history of the Segal-Bargmann transforms for $\mathbb{R}^n$ can be found in [13] and [14].

For infinite dimensional analysis one is forced to consider the heat kernel transform defined on $L^2(\mathbb{R}^n, \mu_t)$. The reason is, that the heat kernel measure forms a projective family of probability measures on $\mathbb{R}^n$ and hence $\{L^2(\mathbb{R}^n, \mu_t)\}_{n\in\mathbb{N}}$ forms a direct and projective family of Hilbert spaces. Similarly, $\{\mathcal{F}_t(\mathbb{C}^n)\}_{n\in\mathbb{N}}$ forms a direct and projective family

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of Hilbert spaces and the $\{H_{t,n} : L^2(\mathbb{R}^n, \mu_t) \to \mathcal{F}_t(\mathbb{C}^n)\}$ is direct and defines a unitary isomorphism $\varinjlim L^2(\mathbb{R}^n, \mu_t) \to \varinjlim \mathcal{F}_t(\mathbb{C}^n)$.

The symmetric spaces of compact and noncompact type form a natural settings for generalizations of the heat kernel transform and the Segal-Bargmann transform. This was first done in [12] where the Segal-Bargmann transforms were extended to the compact group case and compact homogeneous spaces U/K. As in the flat case, the Segal-Bargmann transform H_t on $L^2(U/K)$ is given by the holomorphic extension of $f * h_t$ to the complexification $U_{\mathbb{C}}$ of U. The author showed that the Segal-Bargmann transform is an unitary isomorphism from $L^2(U)$ onto $\mathcal{O}(U_{\mathbb{C}}) \cap L^2(U_{\mathbb{C}}, \nu_t)$, where $\mathcal{O}(U_{\mathbb{C}})$ denotes the space of holomorphic functions on $U_{\mathbb{C}}$ and ν_t is the U-average heat kernel on $U_{\mathbb{C}}$. Analogous results for compact symmetric spaces were given by Stenzel in [37]. The image of the Segal-Bargmann transform H_t is a L^2-Hilbert space of holomorphic functions on the complexification $U_{\mathbb{C}}/K_{\mathbb{C}}$ of U/K. In [12] the heat kernel measure on $U_{\mathbb{C}}/U$ was used to define the Fock space, whereas [37] uses the heat kernel measure on the non-compact dual G/K of U/K. Both measures coincide as can be shown by using the Flensted-Jensen duality [7]. In [20] and [41] the unitarity of the Segal-Bargmann transform was proved using the restriction principle introduced in [30].

Some work has been done on constructing a heat kernel measure on the direct limit of some complex groups. In [11], Gordina constructed the Fock space on $\mathrm{SO}(\infty, \mathbb{C})$, using the heat kernel measure determined by an inner product on the Lie algebra $\mathfrak{so}(\infty, \mathbb{C})$. Another direction is taken in [17] where the Segal-Bargmann transform on path-groups is considered.

In the noncompact case new technical problems arise. In particular, in the compact case, every eigenfunction of the algebra of invariant differential operators as well as the heat kernel itself, extends to a holomorphic function on $U_{\mathbb{C}}/K_{\mathbb{C}}$. This follows from the fact, that each irreducible representation of U extends to a holomorphic representation of $U_{\mathbb{C}}$ with a well understood growth. In the noncompact case this does not hold anymore. The natural complexification in this case is the *Akhiezer-Gindikin domain* $\Xi \subset G_{\mathbb{C}}/K_{\mathbb{C}}$, see [1]. Using results from [21] it was shown in [22] that the image of the Segal-Bargmann transform on G/K can be identified as a Hilbert-space of holomorphic functions on Ξ, but in this case the norm on the Fock space is not given by a density function as in the flat case. Some special cases have also been considered in [15, 16]. A different description that also works for arbitrary positive multiplicity functions was given in [27].

From the point of view of infinite dimensional analysis, the drawback of all of those articles is that only the invariant measure on G/K is considered, so far no description of the image of the space $L^2(G/K, \mu_t)$ under the holomorphic extension of $f * h_t$ exists, except one can describe the space in terms of its reproducing kernel.

The first step to consider the limit of noncompact symmetric spaces was done in [35]. There it was shown for a special class of symmetric spaces G_n/K_n that $\{L^2(G_n/K_n, \mu_t)\}_n$

forms projective family of Hilbert spaces. But no attempt was made to consider the Segal-Bargmann transform.

Our main goal in this article is to use some ideas from the work of J. Wolf, in particular [44], to construct the Segal-Bargmann on limits of special classes of compact symmetric spaces. In [31] the authors introduced the concept of *propagation* of symmetric spaces. The results of [44] applies to this situation resulting in an isometric embedding $\gamma_{n,m}$ of $L^2(U_n/K_n)$ into $L^2(U_m/K_m)$ for $m > n$. Let $M_n = U_n/K_n$, and $M_{n\mathbb{C}} = U_{n\mathbb{C}}/K_{n\mathbb{C}}$. We show, using the ideas from [44] that we have an isometric embedding $\delta_{n,m} : \mathcal{H}_t(M_{\mathbb{C}n}) \to \mathcal{H}_t(M_{\mathbb{C}m})$ such that $H_{t,m} \circ \gamma_{n,m} = \delta_{n,m} \circ H_{t,n}$. This then results in a unitary isomorphism

$$H_{t,\infty} : \varinjlim L^2(M_n) \to \varinjlim \mathcal{H}_t(M_{n\mathbb{C}})\,.$$

In the K_n-invariant case, in general $j_{n,m}(L^2(M_n)^{K_n}) \not\subseteq L^2(M_m)^{K_m}$ for $m > n$. So different maps have to be considered in this case. In this article, we define an isometric embedding $\eta_{n,m} : L^2(M_n)^{K_n} \to L^2(M_m)^{K_m}$ and similarly for $\mathcal{H}_t(M_{n\mathbb{C}})^{K_n}$ with the embeddings $\phi_{n,m}$ such that $H_{t,m} \circ \eta_{n,m} = \phi_{n,m} \circ H_{t,n}$ resulting in a unitary isomorphism of the directed limits. This is the result from [41].

The article is organized as follows. In Section 1 we introduce the basic notation used in this article. In Section 2 we discuss needed results from representation theory and Fourier analysis related to symmetric spaces. The Fock space $\mathcal{H}_t(M)$ is introduced in Section 3. We show that $\mathcal{H}_t(M)$ is a reproducing kernel Hilbert space and determine its reproducing kernel. We also describe $\mathcal{H}_t(M)$ as a sequence space. The Segal-Bargmann transform is introduced in Section 4 and we show that it is an unitary U-isomorphism in Theorem 4.3. In Section 5 we recall the notion of propagation of symmetric spaces introduced in [31]. The infinite limit is considered in the last two sections, Section 6 and Section 7.

1. Basic Notations

Let M be a *symmetric space of the compact type*. Thus, there exists a connected compact semisimple Lie group U and a nontrivial involution $\theta : U \to U$ such that $U_0^\theta \subseteq K \subseteq U^\theta$ and $M = U/K$. Here, $U^\theta = \{u \in U \mid \theta(u) = u\}$ denotes the subgroup of θ-fixed points, and the index $_0$ stands for the connected component containing the identity.

To simplify the exposition we assume that U is *simply connected*. In this case U^θ is connected and hence $K = U^\theta$ is connected and U/K is simply connected. The more general case can be treated by following the ideas in Section 2 in [29].

The base point $eK \in M$ will be denoted by o. Denote the Lie algebra of U by $\mathfrak{u}$. θ defines a Lie algebra involution on $\mathfrak{u}$ which we will also denote by θ. Then $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{s}$, where $\mathfrak{k} = \{X \in \mathfrak{u} \mid \theta(X) = X\}$ and $\mathfrak{s} = \{X \in \mathfrak{u} \mid \theta(X) = -X\}$. Note that $\mathfrak{k}$ is the Lie

algebra of K, $\mathfrak{s} \simeq_K T_o(M)$ via the map $X \mapsto D_X$,

$$D_X f(o) = \left.\frac{d}{dt}\right|_{t=0} f(\exp(tX) \cdot o)$$

and $T(M) \simeq U \times_{\mathrm{Ad}|_{\mathfrak{s}}} \mathfrak{s}$.

As U is compact, there is a faithful representation of U, so we can–and will–assume that U is linear: $U \subseteq \mathrm{U}(n) \subset \mathrm{GL}(n, \mathbb{C})$ for some $n \in \mathbb{N}$. Then $\mathfrak{u} \subseteq \mathfrak{u}(n)$. Define a U-invariant inner product on $\mathfrak{u}$ by

$$\langle X, Y \rangle = -\mathrm{Tr}\, XY = \mathrm{Tr}\, XY^* .$$

By restriction, this defines a K-invariant inner product on $\mathfrak{s}$ and hence a U-invariant metric on M. We note that $\mathfrak{k}$ and $\mathfrak{s}$ are orthogonal subspaces of $\mathfrak{u}$ with respect to $\langle\,\cdot\,,\,\cdot\,\rangle$.

The inner product on $\mathfrak{u}$ determines an inner product on the dual space $\mathfrak{u}^*$ in a canonical way. Furthermore, these inner products extend to the inner products on the corresponding complexifications $\mathfrak{u}_{\mathbb{C}}$ and $\mathfrak{u}_{\mathbb{C}}^*$. All these bilinear forms are denoted by the same symbol $\langle\,\cdot\,,\,\cdot\,\rangle$.

Let $\mathfrak{a} \subseteq \mathfrak{s}$ be a maximal abelian subspace of $\mathfrak{s}$. View $\mathfrak{a}_{\mathbb{C}}^*$ as the space of $\mathbb{C}$-linear maps $\mathfrak{a}_{\mathbb{C}} \to \mathbb{C}$. Then $\mathfrak{a}^* = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \lambda(\mathfrak{a}) \subseteq \mathbb{R}\}$ and $i\mathfrak{a}^* = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \lambda(\mathfrak{a}) \subseteq i\mathbb{R}\}$.

For $\alpha \in \mathfrak{a}_{\mathbb{C}}^*$, let $\mathfrak{u}_{\mathbb{C}} = \{X \in \mathfrak{u}_{\mathbb{C}} \mid (\forall H \in \mathfrak{a}_{\mathbb{C}})\ [H, X] = \alpha(H)X\}$. If $\mathfrak{u}_{\mathbb{C}\alpha} \neq \{0\}$ then $\alpha \in i\mathfrak{a}^*$ and $\mathfrak{u}_{\mathbb{C}\alpha} \cap \mathfrak{u} = \{0\}$. If $\mathfrak{u}_{\mathbb{C}\alpha} \neq \{0\}$, then α is called a *restricted root*. Denote by $\Sigma = \Sigma(\mathfrak{u}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}}) \subset i\mathfrak{a}^*$ the set of restricted roots. Then

$$\mathfrak{u}_{\mathbb{C}} = \mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{u}_{\mathbb{C}\alpha}$$

where $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$.

The simply connected group U is contained as a maximal compact subgroup in the simply connected complex Lie group $U_{\mathbb{C}} \subseteq \mathrm{GL}(n, \mathbb{C})$ with Lie algebra $\mathfrak{u}_{\mathbb{C}} = \mathfrak{u} \otimes_{\mathbb{R}} \mathbb{C}$. Denote by $\theta : U_{\mathbb{C}} \to U_{\mathbb{C}}$ the holomorphic extension of θ. Let $\sigma : U_{\mathbb{C}} \to U_{\mathbb{C}}$ be the conjugation on $U_{\mathbb{C}}$ with respect to U. Thus the derivative of σ is given by $X + iY \mapsto X - iY$, $X, Y \in \mathfrak{u}$. σ is the Cartan involution on $U_{\mathbb{C}}$ with $U = U_{\mathbb{C}}^{\sigma}$. We will also write $\overline{g} = \sigma(g)$ for $g \in U_{\mathbb{C}}$.

Let $K_{\mathbb{C}} = U_{\mathbb{C}}^{\theta}$. Then $K_{\mathbb{C}}$ has Lie algebra $\mathfrak{k}_{\mathbb{C}}$ and K is a maximal compact subgroup of $K_{\mathbb{C}}$. $K_{\mathbb{C}}$ is connected as $U_{\mathbb{C}}$ is simply connected and $M_{\mathbb{C}} = U_{\mathbb{C}}/K_{\mathbb{C}}$ is a simply connected complex symmetric space. As $\sigma(K_{\mathbb{C}}) = K_{\mathbb{C}}$ it follows that σ defines a conjugation on $M_{\mathbb{C}}$ with $(M_{\mathbb{C}}^{\sigma})_o = M$. Thus M is a totally real submanifold of $M_{\mathbb{C}}$. In particular,

Lemma 1.1. *If $F \in \mathcal{O}(M_{\mathbb{C}})$ and $F|_M = 0$, then $F = 0$.*

Let $\mathfrak{g} = \mathfrak{k} + i\mathfrak{s} = \mathfrak{u}_{\mathbb{C}}^{\theta\sigma}$ and let $G = U_{\mathbb{C}}^{\theta\sigma}$ denote the analytic subgroup of $U_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}$. $M^d = G/K$ is a symmetric space of the noncompact type and $M^d = (M_{\mathbb{C}}^{\sigma\theta})_o$. Hence, M^d is also a totally real submanifold of $U_{\mathbb{C}}/K_{\mathbb{C}}$. M^d is called the *noncompact dual* of M.

The following is clear (and well known) using the Cartan decomposition of $U_{\mathbb{C}}$ and G:

Lemma 1.2. *Let $g \in U_{\mathbb{C}}$. Then there exists a unique $u \in U$ and a unique $X \in i\mathfrak{u}$ such that $g = u \exp X$. We have $g \in G$ if and only if $u \in K$ and $X \in i\mathfrak{s}$.*

2. L^2 Fourier Analysis

In this section, we give a brief overview of the representation theory related to harmonic analysis on M.

Since U is assumed to be simply connected, there is a one-to-one correspondence between $\widehat{U}$, the set of equivalence classes of irreducible unitary representations of U, and the semi-lattice of *dominant algebraically integral weights* on a Cartan subalgebra containing $\mathfrak{a}$. We denote this correspondence by $\mu \leftrightarrow (\pi_\mu, V_\mu)$. (π_μ, V_μ) is *spherical* if

$$V_\mu^K = \{v \in V_\mu \mid (\forall k \in K)\ \pi_\mu(k)v = v\} \neq \{0\}\,.$$

There exists an isometric U-intertwining operator $V_\mu \hookrightarrow L^2(M)$ if and only if $V_\mu^K \neq \{0\}$. In that case $\dim V_\mu^K = 1$. The description of the highest weights of the spherical representations is given by the Cartan-Helgason theorem, see Theorem 4.1, p. 535 in [19]. Fix a positive system $\Sigma^+ \subset \Sigma$. Let

$$\Lambda_K^+(U) = \left\{ \mu \in i\mathfrak{a}^* \;\middle|\; (\forall \alpha \in \Sigma^+)\ \frac{\langle \alpha, \mu\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{Z}^+ = \{0, 1, \dots\} \right\}. \tag{2.1}$$

As both U and K will be fixed for the moment we simply write Λ^+ for $\Lambda_K^+(U)$. Λ^+ is contained in the semi-lattice of dominant algebraically integral weights.

Theorem 2.1. *Let (π_μ, V_μ) be an irreducible representation of U with highest weight μ. Then π_μ is spherical if and only if $\mu \in \Lambda^+$.*

If nothing else is said, then we will from now on assume that (π_μ, V_μ) is spherical. $\langle\,\cdot\,,\,\cdot\,\rangle_\mu$ will denote a U-invariant inner product on V_μ. The corresponding norm is denoted by $\|\cdot\|_\mu$. Let $d(\mu) = \dim V_\mu$. Then $\mu \mapsto d(\mu)$ extends to a polynomial function on $\mathfrak{a}_{\mathbb{C}}^*$ of degree $\sum_{\alpha\in\Sigma^+} \dim_{\mathbb{C}} \mathfrak{u}_{\mathbb{C}\alpha}$. We fix $e_\mu \in V_\mu^K$ with $\|e_\mu\| = 1$. The function $g \mapsto \pi_\mu(g)e_\mu$ is right K-invariant and defines a V_μ valued function on M. We write $\pi_\mu(x)e_\mu = \pi_\mu(g)e_\mu$ if $x = g \cdot o$, $g \in U$.

For $u \in V_\mu$ let

$$\pi_\mu^u(x) = \langle u, \pi_\mu(x)e_\mu\rangle_\mu\,. \tag{2.2}$$

The representation π_μ extends to a holomorphic representation of $U_{\mathbb{C}}$ which we will also denote by π_μ. As

$$\pi_\mu(g)^* = \pi_\mu(\sigma(g)^{-1}) \tag{2.3}$$

on U and both sides are anti-holomorphic on $U_{\mathbb{C}}$, it follows that (2.3) holds for all $g \in U_{\mathbb{C}}$. We extend π_μ^u to a holomorphic function on $M_{\mathbb{C}}$ by

$$\widetilde{\pi}_\mu^u(z) = \langle u, \pi_\mu(\sigma(z))e_\mu\rangle_\mu := \langle u, \pi_\mu(\sigma(g))e_\mu\rangle_\mu = \langle \pi_\mu(g^{-1})u, e_\mu\rangle_\mu\,, \quad z = g\cdot o\,. \tag{2.4}$$

We normalize the invariant measure on compact groups so that the total measure of the group is one. Then $\int_M f(m)\,dm = \int_U f(a\cdot o)\,da$ defines a normalized U-invariant measure on M. The corresponding L^2-inner product, respectively norm, is denoted by $\langle\,\cdot\,,\,\cdot\,\rangle_2$, respectively $\|\,\cdot\,\|_2$.

Recall that by Schur's orthogonality relations we have

$$\int_U \langle u, \pi_\mu(g)v\rangle_\mu \langle \pi_\delta(g)x, y\rangle_\delta\,du = \delta_{\mu,\nu}\frac{1}{d(\mu)}\langle u, x\rangle_\mu\langle y, v\rangle_\mu\,. \tag{2.5}$$

In particular, $V_\mu \to L^2(M)$, $u \mapsto d(\mu)^{1/2}\pi_\mu^u$ is a unitary U-isomorphism onto its image $L^2(M)_\mu \subset L^2(M)$. Furthermore

$$L^2(M) = \bigoplus_{\mu\in\Lambda^+} L^2(M)_\mu\,.$$

Furthermore, each function $f \in L^2(M)_\mu$ has a holomorphic extension $\widetilde{f}$ to $M_{\mathbb{C}}$.

Lemma 2.2. *Let the notations be as above.*

(1) *Let $\mu, \delta \in \Lambda^+$, $u \in V_\mu$, $v \in V_\delta$, and $H_1, H_2 \in \mathfrak{a}_{\mathbb{C}}$. Then*

$$\begin{aligned}\int_U \widetilde{\pi}_\mu^u(g\exp H_1)\overline{\widetilde{\pi}_\delta^v(g\exp H_2)}\,dg &= \frac{\delta_{\mu,\delta}}{d(\mu)}\langle u, v\rangle_\mu\langle \pi_\mu(\exp H_2)e_\mu, \pi_\mu(\exp H_1)e_\mu\rangle_\mu\\ &= \frac{\delta_{\mu,\delta}}{d(\mu)}\langle u, v\rangle_\mu\langle e_\mu, \pi_\mu(\exp(H_1 - \sigma(H_2))e_\mu\rangle_\mu.\end{aligned}$$

(2) *Let $L \subset M_{\mathbb{C}}$ be compact. Then there exists a constant $C_L > 0$ such that*

$$|\pi_\mu^u(z)| \le e^{C_L\|\mu\|}\|u\|_\mu$$

for all $z \in L$.

Proof. (1) This follows from Schur's orthogonality relations (2.5):

$$\begin{aligned}\int_U \widetilde{\pi}_\mu^u(g\exp H_1)\overline{\widetilde{\pi}_\delta^v(g\exp H_2)}\,dg &= \int_U \langle u, \pi_\mu(g)\pi_\mu(\exp H_1)e_\mu\rangle_\mu\langle \pi_\delta(g)\pi_\delta(\exp H_2)e_\delta, v\rangle_\delta\,dg\\ &= \frac{\delta_{\mu,\delta}}{d(\mu)}\langle u, v\rangle_\mu\langle \pi_\mu(\exp H_2)e_\mu, \pi_\mu(\exp H_1)e_\mu\rangle_\mu\\ &= \frac{\delta_{\mu,\delta}}{d(\mu)}\langle u, v\rangle_\mu\langle e_\mu, \pi_\mu(\exp(H_1 - \sigma(H_2)))e_\mu\rangle_\mu\end{aligned}$$

where we used that $\pi_\mu(\exp H_2)^* = \pi_\mu(\sigma(\exp H_2)^{-1})$. (2) follows from Lemma 3.9 in [4]. $\square$

For $f \in L^2(U) \subset L^1(U)$ let

$$\pi_\mu(f) = \int_U f(g)\pi_\mu(g)\, dg$$

be the integrated representation. Denote by $P_\mu = \int_K \pi_\mu(k)\, dk : V_\mu \to V_\mu^K$ the orthogonal projection $v \mapsto \langle v, e_\mu\rangle e_\mu$. If $f \in L^2(M) = L^2(U)^K$, then $\pi_\mu(f) = \pi_\mu(f)P_\mu$. Define the (vector valued) *Fourier transform* of $f \in L^2(M)$ by

$$\widehat{f}_\mu := \pi_\mu(f)e_\mu\,. \tag{2.6}$$

Denote the left-regular representation of U on $L^2(M)$ by L. Thus $(L(a)f)(x) = f(a^{-1}\cdot x)$. Then

$$\widehat{L(a)f}_\mu = \pi_\mu(a)\widehat{f}_\mu\,. \tag{2.7}$$

To describe the image of the Fourier transform let

$$\begin{aligned}\bigoplus_{\mu\in\Lambda^+,d} V_\mu \;&:=\; \left\{ (v_\mu)_{\mu\in\Lambda^+} \;\middle|\; (\forall \mu \in \Lambda^+)\ v_\mu \in V_\mu \text{ and } \sum_{\mu\in\Lambda^+} d(\mu)\|v_\mu\|_\mu^2 < \infty \right\}\\ &=\; \left\{ v : \Lambda^+ \to \Pi_{\mu\in\Lambda^+}V_\mu \;\middle|\; (\forall \mu \in \Lambda^+)\ v(\mu) \in V_\mu \text{ and } \sum_{\mu\in\Lambda^+} d(\mu)\|v(\mu)\|^2 < \infty \right\}.\end{aligned}$$

Then $\bigoplus_{\mu\in\Lambda^+,d} V_\mu$ is a Hilbert space with the inner product

$$\langle (v_\mu), (w_\mu)\rangle = \sum_{\mu\in\Lambda^+} d(\mu)\langle v_\mu, w_\mu\rangle_\mu\,.$$

The group U acts unitarily on $\bigoplus_{\mu\in\Lambda^+,d} V_\mu$ by $(\widehat{L}(a)(u_\mu))_\mu = (\pi_\mu(a)u_\mu)_\mu$.

Theorem 2.3. *If $f \in L^2(M)$ then $(\widehat{f}_\mu)_\mu \in \bigoplus_{\mu\in\Lambda^+,d} V_\mu$ and $\widehat{\ } : L^2(M) \to \bigoplus_{\mu\in\Lambda^+,\,d} V_\mu$ is a unitary U-isomorphism with inverse*

$$v \mapsto \sum_{\mu\in\Lambda^+} d(\mu)\pi_\mu^{v(\mu)}\,.$$

In particular if $f \in L^2(M)$ then

$$f = \sum_{\mu\in\Lambda^+} d(\mu)\langle \widehat{f}_\mu, \pi_\mu(\,\cdot\,)e_\mu\rangle_\mu = \sum_{\mu\in\Lambda^+} d(\mu)\pi_\mu^{\widehat{f}_\mu} \quad \textit{and} \quad \|f\|_2^2 = \sum_{\mu\in\Lambda^+} d(\mu)\|\widehat{f}_\mu\|_\mu^2 \tag{2.8}$$

where the first sum is taking in $L^2(M)$. If f is smooth, then the sum converges in the C^∞-topology. Furthermore

1. *If $f \in L^2(M)$ then the orthogonal projection of f into $L^2(M)_\mu$ is given by $f_\mu = d(\mu)\langle \widehat{f}_\mu, \pi_\mu(\,\cdot\,)e_\mu\rangle$.*

(2) *f_μ has a holomorphic continuation $\widetilde{f}_\mu$ to $M_{\mathbb{C}}$ which is given by*

$$\widetilde{f}_\mu(z) = d(\mu)\langle \widehat{f}_\mu, \pi_\mu(\bar{z})e_\mu\rangle\,. \tag{2.9}$$

(3) *If $L \subset M_{\mathbb{C}}$ is compact, then there exists a constant $C_L > 0$ such that*

$$\sup_{x\in L} |\widetilde{f}_\mu(x)| \leq d(\mu)e^{C_L\|\mu\|}\|f\|_2\,.$$

Proof. This is well known but we indicate how the statements follows from the general Plancherel formula for compact groups, see [9], p. 134. We have

$$f(x) = \sum_{\mu\in\Lambda^+} d(\mu)\mathrm{Tr}\,(\pi_\mu(x^{-1})\pi_\mu(f)) \quad \text{and} \quad \|f\|_2^2 = \sum_{\mu\in\Lambda^+} d(\mu)\mathrm{Tr}\,(\pi_\mu(f)^*\pi_\mu(f))\,.$$

Extending e_μ to an orthonormal basis for V_μ, it follows from $\pi_\mu(f) = \pi_\mu(f)P_K$ that

$$\mathrm{Tr}\,(\pi_\mu(x^{-1})\pi_\mu(f)) = \langle \pi_\mu(x^{-1})\pi_\mu(f)e_\mu, e_\mu\rangle_\mu = \langle \widehat{f}(\mu), \pi_\mu(x)e_\mu\rangle_\mu$$

and

$$\mathrm{Tr}\,(\pi_\mu(f)^*\pi_\mu(f)) = \langle \pi_\mu(f)^*\pi_\mu(f)e_\mu, e_\mu\rangle_\mu = \langle \pi_\mu(f)e_\mu, \pi_\mu(f)e_\mu\rangle_\mu = \|\widehat{f}(\mu)\|_\mu^2\,.$$

The L^2-part of the theorem follows now from the Plancherel formula for U. The intertwining property is a consequence of (2.7). For the last statement see [38].

That the orthogonal projection $f \mapsto f_\mu$ is given by $f_\mu(x) = d(\mu)\langle \widehat{f}_\mu, \pi_\mu(x)e_\mu\rangle$ follows from (2.8). The last part follows from (2.4) and $\|\widehat{f}_\mu\|_\mu \leq \|f\|_2$. $\square$

The *spherical function* on M associated with μ is the matrix coefficient

$$\psi_\mu(g) = \pi_\mu^{e_\mu}(g) = \langle e_\mu, \pi_\mu(g)e_\mu\rangle_\mu\ ,\ g \in U\,. \tag{2.10}$$

It is independent of the choice of e_μ as long as $\|e_\mu\|_\mu = 1$. We will view ψ_μ as a K-biinvariant function on U or as a K-invariant function on M. ψ_μ is the unique element in $L^2(M)_\mu^K$ which takes the value one at the base point o.

If $f \in L^2(M)^K$ then $\widehat{f}_\mu = \langle \widehat{f}_\mu, e_\mu\rangle_\mu e_\mu \in V_\mu^K$. Furthermore,

$$\langle \widehat{f}_\mu, e_\mu\rangle_\mu = \langle \pi_\mu(f)e_\mu, e_\mu\rangle_\mu = \int_M f(m)\overline{\psi_\mu(m)}\, dm = \int_U f(a\cdot o)\psi_\mu(a^{-1})\, da\,.$$

This motivates the definition of the *spherical Fourier transform* on $L^2(M)^K$ by

$$\widehat{f}(\mu) = \langle f, \psi_\mu\rangle_2\,. \tag{2.11}$$

Define the weighted ℓ^2-space $\ell^2_d(\Lambda^+)$ by

$$\ell^2_d(\Lambda^+) := \left\{ (a_\mu)_{\mu\in\Lambda^+} \;\middle|\; a_\mu \in \mathbb{C} \text{ and } \sum_{\mu\in\Lambda^+} d(\mu)|a_\mu|^2 < \infty \right\}.$$

Then $\ell^2_d(\Lambda^+)$ is a Hilbert space.

Theorem 2.4. *The spherical Fourier transform is a unitary isomorphism of $L^2(M)^K$ onto $\ell^2_d(\Lambda^+)$ with inverse*

$$(a_\mu)_\mu \mapsto \sum_{\mu\in\Lambda^+} d(\mu) a_\mu \psi_\mu$$

where the sum is taken in $L^2(M)^K$. It converges in the C^∞-topology if f is smooth. Furthermore,

$$\|f\|_2^2 = \sum_{\mu\in\Lambda^+} d(\mu)|\widehat{f}(\mu)|^2 \,.$$

Proof. This follows directly from Theorem 2.3. □

Note that ψ_μ has a holomorphic extension $\widetilde{\psi}_\mu$ to $M_\mathbb{C}$ given by $\widetilde{\psi}_\mu(z) = \langle e_\mu, \pi_\mu(\sigma(z))e_\mu\rangle_\mu$.

Lemma 2.5. *Let $f\in L^2(M)$. Then $f_\mu(z) = d(\mu) f * \widetilde{\psi}_\mu(z)$.*

Proof. We have

$$\begin{aligned} f_\mu(z) &= d(\mu)\langle \widehat{f}_\mu, \pi_\mu(\sigma(z))e_\mu\rangle_\mu \\ &= \int_U f(g\cdot o)\langle \pi_\mu(g)e_\mu, \pi_\mu(\sigma(z))e_\mu\rangle_\mu \, dg \\ &= \int_U f(g\cdot o)\langle e_\mu, \pi_\mu(\sigma(g^{-1}z))e_\mu\rangle_\mu \, dg \\ &= f * \widetilde{\psi}_\mu(z)\,. \end{aligned}$$

□

3. The Fock Space $\mathcal{H}_t(M_\mathbb{C})$

In this section, we start by recalling some needed and well-known facts on integration on $M^d = G/K$, the noncompact dual of M. We then introduce the heat kernel h_t^d on M^d. For more details and proofs for the statements involving h_t^d we refer to [21, 22, 27, 28] and the references therein. We introduce the Fock space $\mathcal{H}_t(M_\mathbb{C})$. Using the restriction principle introduced in [30] we show that $\mathcal{H}_t(M_\mathbb{C})$ is isomorphic to $L^2(M)$ as a U-representation. In the next section we will show that the Segal-Bargmann transform $H_t : L^2(M) \to \mathcal{H}_t(M_\mathbb{C})$ is a unitary isomorphism.

Let

$$(i\mathfrak{a})_+ = \{H\in i\mathfrak{a} \mid (\forall \alpha\in\Sigma^+)\ \alpha(H) > 0\}\,.$$

The following is a well known decomposition theorem for an involution commuting with a given Cartan involution, see [8] or Proposition 7.1.3. in [36].

Lemma 3.1. *Let $z \in M_{\mathbb{C}}$. Then there exist $u \in U$ and $H \in i\mathfrak{a}$ such that $z = u\exp(H)\cdot o$. If $u_1\exp(H_1)\cdot o = u_2\exp(H_2)\cdot o$ then there exists $w \in W$ such that $H_2 = w\cdot H_1$. If we choose $H \in \overline{(i\mathfrak{a})_+}$, then H is unique.*

Let m be a $U_{\mathbb{C}}$-invariant measure on $M_{\mathbb{C}}$.

Theorem 3.2. *We can normalize m such that for $f \in L^1(M_{\mathbb{C}})$*
$$\int_{M_{\mathbb{C}}} f(z)dm(z) = \int_U \int_{(i\mathfrak{a})_+} f(u\exp H\cdot o)J(H)dHdu,$$
where
$$J(H) = \prod_{\alpha\in\Sigma^+} \sinh\left(2\left\langle \alpha, H\right\rangle\right)\,.$$

Proof. This follows from the general integration theorem for symmetric space applied to $M_{\mathbb{C}}$, see [8] or Proposition 8.1.1 in [36], using that $\sinh(2x) = 2\sinh(x)\cosh(x)$. □

Let m_1 be a G-invariant measure on M^d.

Theorem 3.3. *We can normalize m_1 such that for $f \in L^1(M)$*
$$\int_{M^d} f(x)dm_1(x) = \int_K \int_{(i\mathfrak{a})_+} f(k\exp H\cdot o)J_1(H)dHdk\,.$$
where $J_1(H) = J(2^{-1}H)$.

Corresponding to the positive system Σ^+ there is an Iwazawa decomposition $G = KA^dN$ of G, where $A^d = \exp(i\mathfrak{a})$. Write $x \in G$ as $x = k(x)a(x)n(x)$. For $\alpha \in \Sigma$ let $m_\alpha = \dim_{\mathbb{C}} \mathfrak{u}_{\mathbb{C},\alpha}$ and let $\rho = 2^{-1}\sum_{\alpha\in\Sigma^+} m_\alpha\alpha \in i\mathfrak{a}^*$. Let
$$\varphi_\lambda(x) = \int_K a(gk)^{i\lambda-\rho}\,dk$$
denote the spherical functions on M^d with spectral parameter λ, see [19], Theorem 4.3, p. 418, and p. 435.

Lemma 3.4. *Let $\mu \in \Lambda^+$. Then $\varphi_{i(\mu+\rho)}$ extends to a holomorphic function $\widetilde{\varphi}_{i(\mu+\rho)}$ on $M_{\mathbb{C}}$ and $\widetilde{\psi_\mu} = \widetilde{\varphi}_{i(\mu+\rho)}$.*

Proof. See the proof of Lemma 2.5 in [4] and the fact that $\varphi_\lambda(g^{-1}) = \varphi_{-\lambda}(g)$, see [19], p. 419. □

Consider the complex-bilinear extension $(\,\cdot\,,\,\cdot\,)$ of $\langle\,\cdot\,,\,\cdot\,\rangle|_{(i\mathfrak{a})^*\times(i\mathfrak{a})^*}$ to $\mathfrak{a}_{\mathbb{C}}^*$. We write $\lambda\cdot\mu = (\lambda,\mu)$ and $\lambda^2 = (\lambda,\lambda)$.

The trace form $(X,Y) = -\mathrm{Tr}\,(XY^*)$ defines a K-invariant metric on $i\mathfrak{s}$ and hence a G^d-invariant metric on M^d. We consider the Laplace operator Δ^d associated with this metric.

Let h_t^d be the heat kernel associated to the Laplace-Beltrami operator on the noncompact symmetric space M^d. Then h_t^d is K-invariant. Thus $h_t \geq 0$, $\{h_t^d\}_{t>0}$ is an approximate unity and $e^{t\Delta_d} f = h_t * f$. In particular $h_t * \varphi_\lambda = e^{-t(\lambda^2+\rho^2)}\varphi$ for every bounded spherical function.

Theorem 3.5. *Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Then $h_{2t}\varphi_{-\lambda} \in L^1(M^d)$, and*

$$\int_{M^d} h_{2t}^d(x)\varphi_{-\lambda}(x)\, dm(x) = \int_{(i\mathfrak{a})_+} h_{2t}^d(\exp H)\varphi_{-\lambda}(\exp H)J_1(H)\, dH = e^{-2t(\lambda^2+\rho^2)}\,.$$

In particular

$$\begin{aligned}\int_{(i\mathfrak{a})_+} h_{2t}^d(\exp H)\widetilde{\psi}_\mu(\exp H)J_1(H)\, dH &= \frac{1}{|W|}\int_{i\mathfrak{a}} h_{2t}^d(\exp H)\widetilde{\psi}_\mu(\exp H)|J_1(H)|\, dH \\ &= e^{2t(\mu^2+2\mu\cdot\rho)}\,. \qquad (3.1)\end{aligned}$$

Proof. First note that for $H \in (i\mathfrak{a})_+$, we have

$$J_1(H) \leq C_1 e^{2\rho(H)}$$

and by simple use of the estimates for $\varphi_{-\lambda}(\exp H)$ from Proposition 6.1 in [33], we have

$$|\varphi_{-\lambda}(\exp H)| \leq Ce^{\|\lambda\|\|H\|}\,.$$

Finally, according to the Main Theorem in [2], p. 33, there exists a positive polynomial $p(H)$ on $i\mathfrak{a}$ such that

$$h_{2t}^d(\exp H) \leq p(H)e^{-\rho(H)}e^{-\|H\|^2/8t} \qquad \text{for all } H \in (i\mathfrak{a})_+\,.$$

Let $L \subset \mathfrak{a}_{\mathbb{C}}^*$. Putting those three estimates together we get

$$|h_{2t}^d(\exp H)\varphi_{-\lambda}(\exp H)J_1(H)| \leq C_1|p(H)|e^{C_2\|H\|-\|H\|^2/8t}$$

for some constants $C_1, C_2 > 0$. As the right hand side is integrable it follows that $H \mapsto h_{2t}^d(\exp H)\varphi_{-\lambda}(\exp H)J_1(H)$ is integrable on $(i\mathfrak{a})_+$ and

$$\lambda \mapsto \int_{(i\mathfrak{a})_+} h_{2t}^d(\exp H)\varphi_{-\lambda}(\exp H)J_1(H)\, dH = \int_{M^d} h_{2t}^d(m)\varphi_{-\lambda}(m)\, dm$$

is holomorphic.

It is well known, see [2], that for $\lambda \in (i\mathfrak{a})^*$,

$$\int_{M^d} h_{2t}^d(x)\varphi_{-\lambda}(x)dm_1(x) = e^{-2t((\lambda,\lambda)+(\rho,\rho))}\,. \qquad (3.2)$$

As both sides are holomorphic it follows that (3.2) holds for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. As the holomorphic extension is an even function in λ, (3.1) follows now from Lemma 3.4 by taking $\lambda = i(\mu+\rho)$. $\square$

We define

$$p_t(z) := 2^r h^d_{2t}(\exp(2H)\cdot o) \quad \text{for} \quad z = (u\exp H)\cdot o \in M_{\mathbb{C}}\,,\ u \in U\,,\ H \in i\mathfrak{a}\,. \tag{3.3}$$

Here $r = \dim \mathfrak{a}$. Define a measure μ_t on $M_{\mathbb{C}}$ by $d\mu_t(z) := p_t(z)dm(z)$. We note that p_t is U-invariant by definition. As m_1 is $M_{\mathbb{C}}$ invariant, it follows that the measure μ_t is U-invariant. Define the *Fock space* $\mathcal{H}_t(M_{\mathbb{C}})$ by

(3.4)

$$\mathcal{H}_t(M_{\mathbb{C}}) := \left\{ F \in \mathcal{O}(M_{\mathbb{C}}) \;\middle|\; \|F\|_t^2 = \int_{M_{\mathbb{C}}} |F(z)|^2\, d\mu_t(z) < \infty \right\} = L^2(M_{\mathbb{C}}, \mu_t) \cap \mathcal{O}(M_{\mathbb{C}})\,.$$

Using that μ_t is U-invariant we get the following standard results (c.f. [25], [5]):

Theorem 3.6. *Let $t > 0$. Then $\mathcal{H}_t(M_{\mathbb{C}})$ is an U-invariant Hilbert space of holomorphic functions. In particular, if $L \subset M_{\mathbb{C}}$ is compact, then there exists a constant $C_L > 0$ such that*

$$\sup_{z\in L} |F(z)| \le C_L \|F\|_t \quad \textit{for all} \quad F \in \mathcal{H}_t(M_{\mathbb{C}})\,.$$

Furthermore, there exists a function $K_t : M_{\mathbb{C}} \times M_{\mathbb{C}} \to \mathbb{C}$ such that

(1) $K_t(\,\cdot\,, w) \in \mathcal{H}_t(M_{\mathbb{C}})$ *for all* $w \in M_{\mathbb{C}}$.
(2) $F(w) = \langle F, K_t(\cdot, w)\rangle$ *for all* $F \in \mathcal{H}_t(M_{\mathbb{C}})$.
(3) $K_t(z, w) = \overline{K_t(w, z)}$.
(4) $(z, w) \mapsto K_t(z, w)$ *is holomorphic in z and anti-holomorphic in w.*

$K_t(z, w)$ is *the reproducing kernel* of $\mathcal{H}_t(M_{\mathbb{C}})$. Recall that the existence of $K_{t,w}(z) := K_t(z, w)$ follows from the fact that the point evaluation $F \mapsto F(w)$ is continuous on $\mathcal{H}_t(M_{\mathbb{C}})$ and hence this evaluation map is given by the inner product with a function $K_{t,w} \in \mathcal{H}_t(M_{\mathbb{C}})$. Then

$$K_t(z, w) = \langle K_{t,w}, K_{t,z}\rangle$$

which clearly implies (2) and (3).

Lemma 3.7. *If $\mu \in \Lambda^+$ and $v \in V_\mu$ then $\widetilde{\pi}^v_\mu \in \mathcal{H}_t(M_{\mathbb{C}})$. Furthermore, if $\delta \in \Lambda^+$, and $w \in V_\delta$, then*

$$\langle \widetilde{\pi}^v_\mu, \widetilde{\pi}^w_\delta\rangle_t = \delta_{\mu,\delta} \frac{e^{2t(\mu^2 + 2\rho\cdot))}}{d(\mu)} \langle v, w\rangle_\mu.$$

Proof. We show first that $\widetilde{\pi}^v_\mu \in \mathcal{H}_t$. Clearly, $\widetilde{\pi}^v_\mu$ is holomorphic, so we only have to show that it is square integrable with respect to μ_t. We have by Theorem 3.2 and (1) of Lemma

2.2 that

$$\begin{aligned}
\langle \widetilde{\pi}^v_\mu, \widetilde{\pi}^v_\mu \rangle_t &= \int_{M_\mathbb{C}} |\widetilde{\pi}^v_\mu(z)|^2 \, d\mu_t(z) \\
&= 2^r \int_U \int_{(i\mathfrak{a})_+} |\widetilde{\pi}^v_\mu(g \exp H)|^2 h^d_{2t}(\exp(2H)) J(H) dH \; dg \\
&= 2^r \int_{(i\mathfrak{a})_+} h^d_{2t}(\exp(2H)\cdot o) \left(\int_U \widetilde{\pi}^v_\mu(g\exp H) \overline{\widetilde{\pi}^v_\mu(g \exp H)} \, dg \right) J(H) dH \\
&= \frac{2^r \|v\|^2}{d(\mu)} \int_{(i\mathfrak{a})_+} \widetilde{\psi}_\mu(\exp(2H)) J_1(2H) h^d_{2t}(\exp(2H)\cdot o) \, dH \\
&= \frac{\|v\|^2}{d(\mu)} \int_{(i\mathfrak{a})_+} \widetilde{\psi}_\mu(\exp(H)) J_1(H) h^d_{2t}(\exp(H)\cdot o) \, dH \\
&= \frac{\|v\|^2}{d(\mu)} e^{2t\langle \mu + 2\rho, \mu\rangle} \\
&< \infty
\end{aligned}$$

where the last equality follows from Theorem 3.5.

Now using again Theorem 3.2, Lemma 2.2, Theorem 3.5, and Fubini's theorem we get, by the same arguments, that

$$\langle \widetilde{\pi}^v_\mu, \widetilde{\pi}^w_\delta \rangle_t = \delta_{\mu,\delta} \frac{e^{2t\langle \mu+2\rho,\mu\rangle}}{d(\mu)} \langle u, w\rangle_\mu$$

and the Lemma follows. □

Theorem 3.8. *Let $s, R, S > 0$. Assume that $\mu \in v(\mu) \in V_\mu$ is such that $\|v_\mu\|_\mu \leq Re^{S\|\mu\|}$. Then*

$$F(z) = \sum_{\mu \in \Lambda^+} d(\mu) e^{-s\langle \mu+2\rho,\mu\rangle} \widetilde{\pi}^{v(\mu)}_\mu(z)$$

defines a holomorphic function on $M_\mathbb{C}$. If $L \subset M_\mathbb{C}$ is compact, then there exists $C(L) > 0$, independent of $(v_\mu)_\mu$, such that

$$|F(z)| \leq C(L) R\,. \tag{3.5}$$

Proof. Let $L \subseteq M_\mathbb{C}$ be compact. Let $F_\mu(z) = d(\mu)\langle v(\mu), \pi_\mu(\bar{z}) e_\mu\rangle_\mu$. Then F_μ is holomorphic and

$$|F_\mu(z)| \leq e^{(S+C_L)\|\mu\|} \|u\|_\mu \leq Re^{(S+C_L)\|\mu\|}$$

by Lemma 2.2. As, $\mu \mapsto d(\mu)$ is a polynomial function of degree $\sum_{\alpha\in\Sigma^+} \dim_\mathbb{C} \mathfrak{u}_{\mathbb{C}\alpha}$ and $\langle \mu + 2\rho, \mu\rangle \geq 0$, it follows that the function $\Lambda^+ \to \mathbb{R}^+$,

$$\mu \mapsto d(\mu)(1 + \|\mu\|^2)^k e^{(S+C_L)\|\mu\|} e^{-s(\mu^2 + 2\rho\cdot\mu)}$$

is bounded for each $k \in \mathbb{Z}^+$. Hence, there exists a constant $c(k, L, s)$ independent of μ such that

$$d(\mu)e^{(S+C_L)\|\mu\|}e^{-s(\mu^2+2\rho\cdot\mu)} \leq c(k, L, s)(1 + \|\mu\|^2)^{-k}$$

for all $\mu \in \Lambda^+$. By Lemma 1.3 in [38] (see also in Lemma 5.6.7 in [40]) there exists $k_0 \in \mathbb{N}$ such that $\sum_{\mu\in\Lambda^+}(1 + \|\mu\|^2)^{-k_0}$ converges. Hence

$$\sum_{\mu\in\Lambda^+} d(\mu)e^{-s\langle\mu+\rho,\rho\rangle}|F_\mu(z)| \leq c(k_0, L, s)R \sum_{\mu\in\Lambda^+}(1 + \|\mu\|^2)^{-k_0} \leq C(L)R \tag{3.6}$$

converges uniformly on L, and hence defines a holomorphic function on $M_{\mathbb{C}}$. The estimate (3.5) is (3.6). □

For $z = a \cdot o, w = b \cdot o \in M_{\mathbb{C}}$, write

$$\widetilde{\psi}_\mu(w^*z) := \widetilde{\psi}_\mu(\sigma(b)^{-1}a) = L(\overline{w})\widetilde{\psi}_\mu(z)\,.$$

Then $z, w \mapsto \widetilde{\psi}_\mu(w^*z)$ is well defined, holomorphic in z and anti-holomorphic in w. The above Lemma implies that

$$(z, w) \mapsto \sum_{\mu\in\Lambda^+} d(\mu)e^{-2t\langle\mu+2\rho,\mu\rangle}\widetilde{\psi}_\mu(w^*z) = \sum_{\mu\in\Lambda^+} d(\mu)e^{-2t\langle\mu+2\rho,\mu\rangle}\langle\pi_\mu(\overline{w})e_\mu, \pi_\mu(\overline{z})e_\mu\rangle$$

defines a function on $M_{\mathbb{C}} \times M_{\mathbb{C}}$, holomorphic in z and anti-holomorphic in w.

Denote the unitary representation of U on $\mathcal{H}_t(M_{\mathbb{C}})$ by τ_t. Then we have the following theorem:

Theorem 3.9. *Let $t > 0$ then $(\tau_t, \mathcal{H}_t(M_{\mathbb{C}})) \simeq_U (L, L^2(M))$. Furthermore, the reproducing kernel for $\mathcal{H}_t(M_{\mathbb{C}})$ is given by*

$$K_t(z, w) = \sum_{\mu\in\Lambda^+} d(\mu)e^{-2t\langle\mu+2\rho,\mu\rangle}\widetilde{\psi}_\mu(w^*z)\,.$$

Proof. We use the *Restriction Principle* introduced in [30] (see also [26]) and Lemma 3.7 for the proof. Define $R : \mathcal{H}_t(M_{\mathbb{C}}) \to C^\infty(M)$ by $RF := F|_M$. Then R commutes with the action of U. As M is totally real in $M_{\mathbb{C}}$ it follows that R is injective. As M is compact, we get from Theorem 3.6 that there exists a constant $C_M > 0$ such that $\sup_{x\in M} |RF(x)| \leq C_M\|F\|_t$. Thus

$$\|RF\|_2 \leq C_M\|F\|_t\,.$$

Thus $R : \mathcal{H}_t(M_{\mathbb{C}}) \to L^2(M)$ is a continuous U-intertwining operator.

By Lemma 3.7, we have $L^2(M)_\mu \subset \mathrm{Im}(R)$ for each $\mu \in \Lambda^+$. Thus $\mathrm{Im}(R)$ is dense in $L^2(M)$. Write $R^* = U_t\sqrt{RR^*}$. Then $U_t : L^2(M) \to \mathcal{H}_t(M_{\mathbb{C}})$ is an unitary isomorphism and U-intertwining operator.

For $\mu \in \Lambda^+$ let

$$\mathcal{H}_t(M_{\mathbb{C}})_\mu = \{\widetilde{\pi}^u_\mu \mid u \in V_\mu\} = U_t(L^2(M)_\mu)\,.$$

Then $\mathcal{H}_t(M) = \bigoplus_{\mu\in\Lambda^+}\mathcal{H}_t(M_\mathbb{C})_\mu$. As $\mathcal{H}_t(M_\mathbb{C})_\mu \perp \mathcal{H}_t(M_\mathbb{C})_\delta$ for $\mu\neq\delta$ it follows that $K=\sum_{\mu\in\Lambda^+}K_\mu$ where K_μ is the reproducing kernel for $\mathcal{H}_t(M_\mathbb{C})_\mu$. Now, note that

$$L(\overline{w})\widetilde{\psi}_\mu = \widetilde{\pi}_\mu^{\pi_\mu(\overline{w})e_\mu}(z)\,.$$

Thus Lemma 3.7 implies that

$$d(\mu)e^{-2t\langle\mu+2\rho,\mu\rangle}\langle\widetilde{\pi}^u_\mu, L(\overline{w})\widetilde{\psi}_\mu\rangle_t = \langle u,\pi_\mu(\overline{w})e_\mu\rangle_\mu = \widetilde{\pi}^u_\mu(z)\,.$$

Hence $K_\mu(z,w) = d(\mu)e^{-2t\langle\mu+2\rho,\mu\rangle}\widetilde{\psi}_\mu(w^*z)$ and the theorem follows. □

Theorem 3.10. *Define $k_t : M_\mathbb{C}\to\mathbb{C}$ by*

$$k_t(z) := K_t(z,o) = \sum_{\mu\in\Lambda^+} d(\mu)e^{-2t\langle\mu+2\rho,\mu\rangle}\widetilde{\psi}_\mu(z)\,.$$

Let $a,b\in U_\mathbb{C}$ and $z,w\in M_\mathbb{C}$. Then

(1) $K_t(a\cdot z, b\cdot w) = K_t(b^*a\cdot z, w)$.
(2) $K_t(z,w) = k_t(w^*z)$.
(3) $k_t\in\mathcal{H}_t(M_\mathbb{C})^{K_\mathbb{C}}$.
(4) $\overline{k_t(z)} = k_t(z^*)$.
(5) $k_t|_M$ *is real-valued.*
(6) $k_t(x\cdot o) = k_t(x^{-1}\cdot 0)$ *for all* $x\in U_\mathbb{C}$.

Proof. Everything except (5) and (6) follows from Theorem 3.9. Let $w^*\in W$ be the unique element such that $w^*(\Sigma^+) = -\Sigma^+$. Then $-w^*\Lambda^+ = \Lambda^+$ and if $\mu\in\Lambda^+$ then

$$\psi_{-w^*\mu}(x) = \psi_\mu(x^{-1}) = \overline{\psi_\mu(x)}\,,\quad x\in U. \tag{3.7}$$

This is well known and follows easily from Lemma 3.4: We have

$$\begin{aligned}\psi_{-w^*\mu}(x) &= \widetilde{\varphi}_{i(-w^*\mu+\rho)}(x)\\ &= \widetilde{\varphi}_{-w^*(i(\mu+\rho))}(x)\\ &= \widetilde{\varphi}_{-i(\mu+\rho)}(x)\\ &= \widetilde{\varphi}_{i(\mu+\rho)}(x^{-1})\\ &= \psi_\mu(x^{-1})\\ &= \overline{\psi_\mu(x)}\end{aligned}$$

It follows that $k_t(x) = \frac{1}{2}\sum_{\mu\in\Lambda^+}(\psi_\mu(x)+\psi_{-w^*\mu}(x)) = \sum_{\mu\in\Lambda^+}\mathrm{Re}(\psi_\mu(x))$ and hence $k_t(x)$ is real.

By the same argument, we see that for $x\in U$ we have

$$k_t(x) = \frac{1}{2}\sum_{\mu\in\Lambda^+}(\psi_\mu(x)+\psi_{-w^*\mu}(x)) = \frac{1}{2}\sum_{\mu\in\Lambda^+}(\psi_\mu(x)+\psi_\mu(x^{-1}))$$

so $k_t(x) = k_t(x^{-1})$ on U. But both sides are holomorphic on $U_{\mathbb{C}}$ and therefore agree on $U_{\mathbb{C}}$. $\square$

Define

$$\mathcal{F}_t(\Lambda^+) := \left\{ a : \Lambda^+ \to \prod_{\mu\in\Lambda^+} V_\mu \;\middle|\; a(\mu) \in V_\mu \text{ and } \sum_{\mu\in\Lambda^+} d(\mu) e^{2t\langle\mu+2\rho,\mu\rangle} \|a(\mu)\|_\mu^2 < \infty \right\} .$$

Then $\mathcal{F}_t(\Lambda^+)$ is a Hilbert space with inner product

$$\langle a, b\rangle_{\mathcal{F}} = \sum_{\mu\in\Lambda^+} d(\mu) e^{2t\langle\mu+2\rho,\mu\rangle} \langle a(\mu), b(\mu)\rangle_\mu$$

and U acts unitarily on $\mathcal{F}_t(\Lambda^+)$ by

$$[\sigma_t(x)(a)](\mu) = \pi_\mu(x) a(\mu) .$$

Theorem 3.11. *We have the followings.*

1. *For $\mu \in \Lambda^+$ let $v_\mu^1, \dots, v_\mu^{d(\mu)}$ be an orthonormal basis for V_μ. Let $\widetilde{\pi}_\mu^j = \widetilde{\pi}_\mu^{v_\mu^j}$. Then $\left\{ \sqrt{d(\mu)} e^{-t\langle\mu+2\rho,\mu\rangle}\, \widetilde{\pi}_\mu^j \;\middle|\; \mu \in \Lambda^+ ,\ i = 1, \dots, d(\mu) \right\}$ is an orthonormal basis for $\mathcal{H}_t(M_{\mathbb{C}})$.*
2. *The map $a \mapsto \sum_{\mu\in\Lambda^+} d(\mu) \widetilde{\pi}_\mu^{a(\mu)}$ is a unitary U-isomorphism, $\mathcal{F}_t(\Lambda^+) \simeq \mathcal{H}_t(M_{\mathbb{C}})$.*
3. *The set $\{\sqrt{d(\mu)} e^{-t\langle\mu+2\rho,\mu\rangle}\, \widetilde{\psi}_\mu \mid \mu \in \Lambda^+\}$ is an orthonormal basis for $\mathcal{H}_t(M_{\mathbb{C}})^K$.*
4. *$\mathcal{H}_t(M_{\mathbb{C}})^K$ is isometrically isomorphic to the sequence space*

$$\left\{ a : \Lambda^+ \to \mathbb{C} \;\middle|\; \sum_{\mu\in\Lambda^+} d(\mu) e^{2t\langle\mu+2\rho,\mu\rangle} |a(\mu)|^2 < \infty \right\} .$$

The isomorphism is given by $a \mapsto \sum_{\mu\in\Lambda^+} d(\mu) a(\mu) \widetilde{\psi}_\mu$.

Proof. (1) This follows from Theorem 3.9 and Lemma 3.7.

(2) For $a \in \mathcal{F}_t(\Lambda^+)$ define $F := \sum_{\mu\in\Lambda^+} d(\mu) \widetilde{\pi}_\mu^{a(\mu)}$. Let $v_\mu = e^{t\langle\mu+2\rho,\mu\rangle} a(\mu)$. Then the sequence $\{\|v_\mu\|_\mu\}$ is bounded. As $F = \sum_{\mu\in\Lambda^+} d(\mu) e^{-t\langle\mu+2\rho,\mu\rangle} \widetilde{\pi}_\mu^{v_\mu}$ it follows from Lemma 3.8 that the series converges and that F is holomorphic. Furthermore,

$$\begin{aligned} \|F\|_t^2 &= \sum_{\mu\in\Lambda^+} d(\mu)^2 \|\widetilde{\pi}_\mu^{a(\mu)}\|_t^2 \\ &= \sum_{\mu\in\Lambda^+} d(\mu)^2 e^{2t\langle\mu+2\rho,\mu\rangle} d(\mu)^{-1} \|a(\mu)\|_\mu^2 \\ &= \sum_{\mu\in\Lambda^+} d(\mu) e^{2t\langle\mu+2\rho,\mu\rangle} \|a(\mu)\|_\mu^2 < \infty . \end{aligned}$$

Hence $F \in \mathcal{H}_t(M_\mathbb{C})$ and $a \mapsto F$ is an isometry. Now, let $F \in \mathcal{H}_t(M_\mathbb{C})$. By Theorem 3.9 $F = \sum_\mu d(\mu) e^{-2t\langle \mu+2\rho,\mu\rangle} \langle F, \widetilde{\pi}_\mu^j\rangle_t \widetilde{\pi}_\mu^i$ and $\|F\|^2 = \sum_{\mu\in\Lambda^+} d(\mu) e^{-2t\langle\mu+2\rho,\mu\rangle} |\langle F, \widetilde{\pi}_\mu^j\rangle_t|^2 < \infty$. Letting

$$a(\mu) = \sum_{j=1}^{d(\mu)} e^{-2t\langle\mu+2\rho,\mu\rangle} \langle F, \widetilde{\pi}_\mu^j\rangle_t \, v_\mu^j$$

we get

$$\sum_{\mu\in\Lambda^+} d(\mu) e^{2t\langle\mu+2\rho,\mu\rangle} \|a(\mu)\|^2 = \|F\|_t^2 < \infty$$

and $F = \sum_{\mu\in\Lambda^+} d(\mu)\widetilde{\pi}_\mu^{a(\mu)}$. Hence $a \mapsto F$ is a unitary isomorphism. That this map is an intertwining operator follows from the equation

$$\widetilde{\pi}_\mu^v(x^{-1}y) = \widetilde{\pi}_\mu^{\mu_\mu(x)v}(y)\,.$$

(3) and (4) now follows as $\widetilde{\psi}_\mu = \widetilde{\pi}_\mu^{e_\mu}$. □

4. Segal-Bargmann Transforms on $L^2(M)$ and $L^2(M)^K$

In this section, we introduce the heat equation and the heat semigroup $e^{t\Delta}$. We show that if $f \in L^2(M)$ then $e^{t\Delta} f$ extends to the holomorphic function $H_t f$ on $M_\mathbb{C}$ and $H_t f \in \mathcal{H}_t(M_\mathbb{C})$. Then we show that the map $H_t : L^2(M) \to \mathcal{H}_t(M_\mathbb{C})$, $H_t(f) = H_t f$, is the unitary isomorphism U_t in the proof of Theorem 3.9. The isomorphism $H_t : L^2(M) \to \mathcal{H}_t(M_\mathbb{C})$ was first established in [12] and [37]. A different proof was later given by Faraut in [5] using Gutzmer's formula. In [20] the Restriction Principle was used. Our proof is also based on the Restriction Principle and uses some ideas from [5]. The K-invariant case was treated in Chapter 4 of [41].

The heat equation on M is the Cauchy problem

$$\begin{aligned} \Delta u(x,t) &= \frac{\partial u}{\partial t}(x,t)\,, \quad (x,t) \in M \times (0,\infty) \\ \lim_{t\to 0^+} u(x,t) &= f(x)\,, \quad f \in L^2(M) \text{ (the initial condition)} \end{aligned}$$

where Δ is the Laplace-Beltrami operator on M defined by $\langle\cdot,\cdot\rangle$. Δ is a self adjoint negative operator on M and a solution to the heat equation is give by the *heat semigroup* $e^{t\Delta}$ applied to f, $u(\cdot,t) = e^{t\Delta} f$.

Lemma 4.1. *Δ acts on $L^2(M)_\mu$ by* $-\langle\mu+2\rho,\mu\rangle\mathrm{Id}$.

Proof. This is well know for the Laplace-Beltrami operator constructed by the Killing form metric. But scaling the inner product on $\mathfrak{s}$ by a constant $c > 0$, results in scaling the Laplace-Beltrami operator as well as the inner product on $\mathfrak{s}^*$ by $1/c$. □

Lemma 4.2. *Let $f \in L^2(M)$. Write $f = \sum_{\mu\in\Lambda^+} f_\mu$ with $f_\mu = d(\mu) f * \widetilde{\psi}_\mu \in L^2(M)_\mu \subset C^\infty(M)$. Then*

$$e^{t\Delta} f = \sum_{\mu\in\Lambda^+} e^{-t\langle\mu+2\rho,\mu\rangle} f_\mu \,. \tag{4.1}$$

Proof. This follows from Lemma 4.1. □

We call the map $H_t : L^2(M) \to \mathcal{O}(M_{\mathbb{C}})$ the *Heat transform* or the *Segal-Bargmann* transform on $L^2(M)$.

Theorem 4.3. *If $f \in L^2(M)$ then $H_t(f) \in \mathcal{H}_t(M_{\mathbb{C}})$ and $H_t : L^2(M) \to \mathcal{H}_t(M_{\mathbb{C}})$ is a unitary U-isomorphism. Furthermore*

1. *Let $h_t = k_{t/2}$. Then $H_t f(z) = (f * h_t)(z)$, see (4.2).*
2. *$H_t = U_t$ where $U_t : L^2(M) \to \mathcal{H}_t(M_{\mathbb{C}})$ is the unitary isomorphism from the proof of Theorem 3.9.*

Proof. We have

$$H_t f = \sum_{\mu\in\Lambda^+} d(\mu) e^{-t\langle\mu+2\rho,\mu\rangle} \widetilde{\pi}_\mu^{\widehat{f}_\mu} = \sum_{\mu\in\Lambda^+} d(\mu) \widetilde{\pi}_\mu^{a(\mu)}$$

with $a(\mu) = e^{-t\langle\mu+2\rho,\mu\rangle} \widehat{f}_\mu$. By Theorem 2.3

$$\sum_{\mu\in\Lambda^+} d(\mu) e^{2t\langle\mu+2\rho,\mu\rangle} \|a(\mu)\|_\mu^2 = \sum_{\mu\in\Lambda^+} d(\mu) \|\widehat{f}_\mu\|_\mu^2 = \|f\|_2^2 \,.$$

By Theorem 3.11, $H_t f$ extends to a holomorphic function on $M_{\mathbb{C}}$, $H_t f \in \mathcal{H}_t(K_{\mathbb{C}})$, and $\|H_t f\|_t = \|f\|_2$. That H_t is bijective follows easily not only from the representation theory but also from the fact, which we will prove in a moment, that $H_t = U_t$.

(1) For $f \in L^1(M)$ and $g \in L^1(M)^K$, or g holomorphic and K-invariant on $M_{\mathbb{C}}$, define

$$(f * g)(m) := \int_U f(x\cdot o) g(x^{-1}\cdot m)\, dx \,. \tag{4.2}$$

We have $|f| * |\widetilde{\psi}_\mu|(z) \le e^{C_U\|\mu\|}\|f\|_2$. Hence $\sum_{\mu\in\Lambda^+} d(\mu) e^{-t\langle\mu+\rho,\mu\rangle} |f| * |\widetilde{\psi}_\mu|(z) < \infty$ and we can interchange the integration and summation to get (with some obvious abuse of notation)

$$f * h_t = \sum_{\mu\in\Lambda^+} d(\mu) e^{-t\langle\mu+2\rho,\mu\rangle} f * \widetilde{\psi}_\mu = \sum_{\mu\in\Lambda^+} e^{-t\langle\mu+2\rho,\mu\rangle} f_\mu = H_t f$$

where the last equality follows from Lemma 4.2.

(2) We use the ideas from [30]. Let R and U_t be as in the proof of Theorem 3.9. Let $f \in L^2(M)$. Then

$$R^* f(z) = \langle R^* f, K_t(\,\cdot\,,z)\rangle_2 = \langle f, RK_t(\,\cdot\,,z)\rangle_2 = \int_M f(x) K_t(z,x)\, dx = \int_M f(x) k_t(x^{-1}z)\, dx \,.$$

In particular

$$RR^*f(m) = f * k_t(m)\,.$$

It follows that $RR^*f = e^{2t\Delta}f$. As $s \mapsto e^{s\Delta}$ is an operator valued semigroup it follows that $\sqrt{RR^*} = e^{t\Delta}$. The image of $H_t = \sqrt{RR^*}$ is dense in $L^2(M)$. Let $g = H_t f \in H_t(L^2(M))$. Then

$$RU_t g = RR^* f = H_t g\,.$$

As $U_t g$ and $H_t g$ are holomorphic and agree on M it follows that $U_t g(z) = H_t g(z)$ on $M_{\mathbb{C}}$. The image of H_t is dense in $L^2(M)$ and both U_t and H_t are continuous, hence $U_t = H_t$. □

Define $F_t : \bigoplus_{\mu\in\Lambda^+,\,d} V_\mu \to \mathcal{F}_t(\Lambda^+)$ by $F_t(v)(\mu) := e^{-t\langle\mu+2\rho,\mu\rangle}v(\mu)$.

Corollary 4.4. *We have a commutative diagram of unitary U-isomorphisms*

$$\begin{tikzcd} L^2(M) \arrow[r, "H_t"] \arrow[d, "\widehat{\ }"'] & \mathcal{H}_t(M_{\mathbb{C}}) \arrow[d, "F\mapsto a"] \\ \bigoplus_{\mu\in\Lambda^+,\,d} V_\mu \arrow[r, "F_t"'] & \mathcal{F}_t(\Lambda^+) \end{tikzcd} .$$

Proof. This follows from (4.1), Theorem 2.3, Theorem 3.11, and Theorem 4.3. □

5. Propagations of Compact Symmetric Spaces

In this section, we describe the results, which we need later on, from [31, 44] on limits of symmetric spaces and the related representation theory and harmonic analysis. We mostly follow the discussion and notations in [31]. Most of the material on spherical representations is taking from Section 6 in [31, 32]. We keep the notations from the previous sections and indicate the dependence on the symmetric spaces by the indices m, n etc. In particular $M_n = U_n/K_n$, $n \in \mathbb{N}$ is a sequence of simply connected symmetric spaces of compact type. We assume that for $m \geq n$, $U_n \subseteq U_m$ and $\theta_m|_{\mathfrak{u}_n} = \theta_n$. Then $K_n = K_m \cap U_n$, $\mathfrak{k}_m \cap \mathfrak{u}_n = \mathfrak{k}_n$, and $\mathfrak{s}_m \cap \mathfrak{u}_n$. We recursively choose maximal commutative subspaces $\mathfrak{a}_m \subset \mathfrak{s}_m$ such that $\mathfrak{a}_n = \mathfrak{a}_m \cap \mathfrak{u}_n$ for all $m \geq n$. Let $r_n = \dim \mathfrak{a}_n$ be the rank of M_n.

As before, we let $\Sigma_n = \Sigma(\mathfrak{u}_{n,\mathbb{C}}, \mathfrak{a}_{n,\mathbb{C}})$ denote the system of restricted roots of $\mathfrak{a}_{n,\mathbb{C}}$ in $\mathfrak{u}_{n,\mathbb{C}}$. We can–and will–choose positive systems so that $\Sigma_n^+ \subseteq \sigma_m^+|_{\mathfrak{a}_n}$. Let

$$\Sigma_{1/2,n} = \left\{\alpha \in \Sigma_n \mid \frac{1}{2}\alpha \notin \Sigma_n\right\} \text{ and } \Sigma_{2,n} = \{\alpha \in \Sigma_n \mid 2\alpha \notin \Sigma_n\}\,.$$

Then $\Sigma_{1/2,n}$ and $\Sigma_{2,n}$ are reduced root systems (see Lemma 3.2, p. 456 in [18]). Consider the positive systems $\Sigma_{1/2,n}^+ := \Sigma_{1/2,n} \cap \Sigma_n^+$ and $\Sigma_{2,n}^+ := \Sigma_{2,n} \cap \Sigma_n^+$. Let $\Psi_{1/2,n}$ and $\Psi_{2,n}$ denote the sets of simple roots for $\Sigma_{1/2,n}^+$ and $\Sigma_{2,n}^+$ respectively.

Suppose for a moment that M_n is an irreducible symmetric space for every n. We say that M_m *propagates* M_n if $\Sigma_{1/2,n} = \Sigma_{1/2,m}$ or we only add simple roots to the left end of the Dynkin diagram for $\Psi_{1/2,n}$ to obtain the Dynkin diagram for $\Psi_{1/2,m}$. In particular, $\Psi_{1/2,n}$ and $\Psi_{1/2,m}$ are of the same type. In general, if $M_m \simeq M_m^1 \times \ldots \times M_m^r$ and $M_n \simeq M_n^1 \times M_n^s$ with M_m^i and M_n^j irreducible, then M_m *propagates* M_n if we can enumerate the irreducible factors M_m^i and M_n^j such that M_m^i propagates M_n^i for $i = 1, 2, ..., s$. We refer to the discussion in Section 1 of [32] for more details.

From now on, we assume that M_m *propagates* M_n for all $m \geq n$. We call the sequence $\{M_n = U_n/K_n\}$, the *propagating sequence* of symmetric spaces of compact type. This includes sequences of symmetric spaces from each line of the following table of classical symmetric spaces, see [19], Table V, page 518:

(5.1)

Compact Irreducible Riemannian Symmetric $M = U/K$				
Type	U	K	rank M	dim M
A_{n-1}	$\mathrm{SU}(n) \times \mathrm{SU}(n)$	$\mathrm{diag}\,\mathrm{SU}(n)$	$n-1$	n^2-1
B_n	$\mathrm{SO}(2n+1) \times \mathrm{SO}(2n+1)$	$\mathrm{diag}\,\mathrm{SO}(2n+1)$	n	$2n^2+n$
C_n	$\mathrm{Sp}(n) \times \mathrm{Sp}(n)$	$\mathrm{diag}\,\mathrm{Sp}(n)$	n	$2n^2+n$
D_n	$\mathrm{SO}(2n) \times \mathrm{SO}(2n)$	$\mathrm{diag}\,\mathrm{SO}(2n)$	n	$2n^2-n$
AI	$\mathrm{SU}(n)$	$\mathrm{SO}(n)$	$n-1$	$\frac{(n-1)(n+2)}{2}$
AII	$\mathrm{SU}(2n)$	$\mathrm{Sp}(n)$	$n-1$	$2n^2-n-1$
$AIII$	$\mathrm{SU}(p+q)$	$\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$	$\min(p,q)$	$2pq$
BDI	$\mathrm{SO}(p+q)$	$\mathrm{SO}(p) \times \mathrm{SO}(q)$	$\min(p,q)$	pq
$DIII$	$\mathrm{SO}(2n)$	$\mathrm{U}(n)$	$[\frac{n}{2}]$	$n(n-1)$
CI	$\mathrm{Sp}(n)$	$\mathrm{U}(n)$	n	$n(n+1)$
CII	$\mathrm{Sp}(p+q)$	$\mathrm{Sp}(p) \times \mathrm{Sp}(q)$	$\min(p,q)$	$4pq$

But we can also include an inclusion like

$$SU(n)/SO(n) \subset (SU(n) \times SU(n))/\mathrm{diag}(SU(n) \times SU(n)).$$

Now, let $\Psi_{2,n} = \{\alpha_{n,1}, \ldots, \alpha_{n,r_n}\}$. According to the root systems discussed in Section 2 of [31], we can choose the ordering so that for $j \leq r_n$, $\alpha_{m,j}$ is the unique element of $\Psi_{2,m}$ whose restriction to $\mathfrak{a}_n$ is $\alpha_{n,j}$. Define $\xi_{n,j} \in i\mathfrak{a}_n^*$ by

$$\frac{\langle \xi_{n,j}, \alpha_{n,i} \rangle}{\langle \alpha_{n,i}, \alpha_{n,i} \rangle} = \delta_{i,j} \tag{5.2}$$

The weights $\xi_{n,j}$ are the *class-1 fundamental weights* for $(\mathfrak{u}_n, \mathfrak{k}_n)$. We set

$$\Xi_n = \{\xi_{1,r_1}, \ldots, \xi_{n,r_n}\}.$$

It is clear by the definition of Λ_n^+ that

$$\Lambda_n^+ = \sum_{j=1}^{r_n} \mathbb{Z}^+ \xi_{n,j}\,. \tag{5.3}$$

Lemma 5.1 ([43], Lemma 6, [31], Lemma 6.7)**.** *Recall the root ordering of* (5.2)*. If $1 \leq j \leq r_n$ then $\xi_{m,j}$ is the unique element of Ξ_m whose restriction of $\mathfrak{a}_n$ is $\xi_{n,j}$.*

This allows us to construct the map $\iota_{n,m} : \Lambda_n^+ \to \Lambda_m^+$

$$\iota_{n,m}\left(\sum_{j=1}^{r_n} k_j \xi_{n,j}\right) := \sum_{j=1}^{r_n} k_j \xi_{m,j}\,. \tag{5.4}$$

Note that $\iota_{n,m}(\mu)|_{\mathfrak{a}_n} = \mu$ and if $\delta \in \Lambda_m^+$ is such that $\delta = \sum_{j=1}^{r_n} k_j \xi_{m,j}$, then $\delta|_{\mathfrak{a}_n} \in \Lambda_n^+$ and $\iota_{n,m}(\delta|_{\mathfrak{a}_n}) = \delta$.

Lemma 5.2 ([31], Lemma 6.8)**.** *Assume that $\delta \in \Lambda_m^+$ is a combination of the first r_n fundamental weights, $\delta = \sum_{j=1}^{r_n} k_j \xi_{m,j}$. Let $\mu := \delta|_{\mathfrak{a}_n} = \sum_{j=1}^{r_n} k_j \xi_{n,j}$. If v_δ is a nonzero highest weight vector in V_δ then $\langle \pi_\delta(U_n) v_\delta \rangle$, the linear span of $\{\pi_\mu(g) v_\delta \mid g \in U_n\}$, is an irreducible representation of U_n which is isomorphic to (π_δ, V_μ). Furthermore, v is a highest weight vector for π_μ and π_μ occurs with multiplicity one in $\pi_\delta|_{G_n}$.*

The point of this discussion is, that if $\mu \in \Lambda_n^+$, then we can–and will–view V_μ as a subspace of $V_{\iota_{n,m}(\mu)}$ such that $\langle u, v \rangle_\mu = \langle u, v \rangle_{\iota_{n,m}(\mu)}$ for all $u, v \in V_\mu$.

We note that the K_n-invariant vector $e_\mu \in V_\mu$ is not necessarily K_m-invariant. But the projection of $e_{\iota_{n,m}(\mu)}$ onto V_μ is always non-zero and K_n invariant. This follows from the Lemma 5.2 and the fact that $\langle v_\delta, e_\delta \rangle_\delta \neq 0$ (see [19], the proof of Theorem 4.1, Chapter V). In particular $e_{\iota_{n,m}(\mu)} = c e_\mu + f_{n,m;\mu}$ for some K_n-fixed vector $f_{n,m;\mu}$, orthogonal to e_μ.

6. The Segal-Bargman Transform on the Direct Limit of $\{L^2(M_n)\}_n$

In this section, we recall the isometric U_n-embedding on $L^2(M_n)$ into $L^2(M_m)$ due to J. Wolf, [44]. We then discuss similar construction for the Fock spaces and show that the Segal-Bargmann transform extends to the Hilbert-space direct limit.

First, define $\gamma_{m,n} : L^2(M_n) \longrightarrow L^2(M_m)$ by

$$\begin{aligned} \gamma_{n,m}(f) &:= \sum_{\mu \in \Lambda_n^+} d((\iota_{n,m}(\mu)) \sqrt{\frac{d(\mu)}{d(\iota_{n,m}(\mu))}}\, \langle \widehat{f}_\mu, \pi_{\iota_{n,m}(\mu)}(\,\cdot\,) e_{\iota_{n,m}(\mu)} \rangle_{\iota_{n,m}(\mu)} \\ &= \sum_{\mu \in \Lambda_n^+} \sqrt{d((\iota_{n,m}(\mu)) d(\mu)}\, \langle \widehat{f}_\mu, \pi_{\iota_{n,m}(\mu)}(\,\cdot\,) e_{\iota_{n,m}(\mu)} \rangle_{\iota_{n,m}(\mu)}\,. \end{aligned}$$

Clearly each $\gamma_{m,n}$ is linear. Then by Theorem 2.3, in particular (2.8) it follows that $\gamma_{n,m}$ is an isometry. (2.7) implies that $\gamma_{n,m}$ is an intertwining operator. Moreover, if $n \leq m \leq p$, then

$$\gamma_{n,p} = \gamma_{m,p} \circ \gamma_{n,m}.$$

Therefore we have a direct system of Hilbert spaces $\{L^2(M_n), \gamma_{m,n}\}$ so the Hilbert space direct limit

$$L^2(M_\infty) := \varinjlim \{L^2(M_n), \gamma_{n,m}\}$$

is well defined. Denote by γ_n the canonical isometric embedding $L^2(M_n) \hookrightarrow L^2(M_\infty)$. As $\gamma_{n,m}$ intertwines L_n and L_m it follows that we have a well defined unitary representation of $U_\infty := \varinjlim U_n$ on $L^2(M_\infty)$ given by: If $x \in U_n$ and $f = \gamma_n(f_n) \in L^2(M_\infty)$, then $L_\infty(x)f = \gamma_n(L_n(x)f_n)$. Then γ_n is a unitary U_n-map and according to [44], $L^2(M_\infty)$ is a multiplicity free representation of U_∞. We skip the details as they will not be needed here.

For simplicity write

$$e_n(t,\mu) := e^{t\langle \mu + 2\rho_n, \mu\rangle}\,, \quad \mu \in \Lambda_n^+\,.$$

Next, we define a isometric embedding $\delta_{n,m} : \mathcal{H}_t(M_{n\mathbb{C}}) \hookrightarrow \mathcal{H}_t(M_{m\mathbb{C}})$ for the Fock spaces using Theorem 3.11 and such that the diagram

$$\begin{tikzcd} L^2(M_n) \arrow[r, "\gamma_{n,m}"] \arrow[d, "H_{t,n}"'] & L^2(M_m) \arrow[d, "H_{t,m}"] \\ \mathcal{H}_t(M_{n\mathbb{C}}) \arrow[r, "\delta_{n,m}"] & \mathcal{H}_t(M_{m\mathbb{C}}) \end{tikzcd} \tag{6.1}$$

commutes. This forces us to define $\delta_{n,m}$ by

$$\delta_{n,m}\left(\sum_{\mu\in\Lambda_n^+} d(\mu)\widetilde{\pi}_\mu^{a(\mu)}\right) := \sum_{\mu\in\Lambda_n^+} d(\iota_{n,m}(\mu))\sqrt{\frac{d(\mu)}{d(\iota_{n,m}(\mu))}}\frac{e_n(t,\mu)}{e_m(t,\iota_{n,m}(\mu))}\widetilde{\pi}^{a(\mu)}_{\iota_{n,m}(\mu)}\,. \tag{6.2}$$

Here we use the notation from Theorem 3.11 and view $V_\mu \subseteq V_{\iota_{n,m}(\mu)}$ so $a(\mu) \in V_{\iota_{n,m}(\mu)}$.

Lemma 6.1. *If $m > n$ then $\delta_{n,m} : \mathcal{H}_t(M_{n\mathbb{C}}) \to \mathcal{H}_t(M_{m\mathbb{C}})$ is an isometric U_n-map and the diagram (6.1) commutes. Furthermore, if $n \leq m \leq p$ then $\delta_{n,p} = \delta_{m,p} \circ \delta_{n,p}$.*

Proof. Write $\nu = \iota_{n,m}(\mu)$. We have

$$\begin{aligned}
\|\delta_{n,m}(\sum_{\mu\in\Lambda_n^+} d(\mu)\widetilde{\pi}_\mu^{a(\mu)})\|_{m,t}^2 &= \sum_{\mu\in\Lambda_n^+} d(\nu)e_m(2t,\nu))\frac{d(\mu)}{d(\nu)}\frac{e_n(2t,\mu)}{e_m(2t,\nu)}\|a(\mu)\|_\nu^2 \\
&= \sum_{\mu\in\Lambda_n^+} d(\mu)e_n(2t,\mu)\|a(\mu)\|_\mu^2 \\
&= \|\sum_{\mu\in\Lambda_n^+} d(\mu)\widetilde{\pi}_\mu^{a(\mu)})\|_{n,t}^2 .
\end{aligned}$$

Theorem 3.11 implies that $\delta_{n,m} : \mathcal{H}_t(M_{n\mathbb{C}}) \to \mathcal{H}_t(M_{m\mathbb{C}})$ is an unitary U-isomorphism onto its image. □

The following is now clear from the universal mapping property of the direct limit of Hilbert spaces, see [24]:

Theorem 6.2. *Let $L^2(M_\infty) := \varinjlim\{L^2(M_n), \gamma_{n,m}\}$ as before and $\mathcal{H}_t(M_{\infty\mathbb{C}}) := \varinjlim \mathcal{H}_t(M_{n\mathbb{C}})$ in the category of Hilbert spaces and isometric embeddings. Then there exists a unique unitary isomorphism $H_{t,\infty} : L^2(M_\infty) \to \mathcal{H}_t(M_{\infty\mathbb{C}})$ such that the diagram*

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & L^2(M_n) \arrow[r, "\gamma_{n+1,n}"] \arrow[d, "H_{t,n}"'] & L^2(M_{n+1}) \arrow[r] \arrow[d, "H_{t,n+1}"] & \cdots \arrow[r] \arrow[d] & L^2(M_\infty) \arrow[d, "H_{t,\infty}"'] \\
\cdots \arrow[r] & \mathcal{H}_t(M_{n\mathbb{C}}) \arrow[r, "\delta_{n+1,n}"] & \mathcal{H}_t(M_{n+1,\mathbb{C}}) \arrow[r] & \cdots \arrow[r] & \mathcal{H}_t(M_{\infty\mathbb{C}})
\end{tikzcd}$$

commutes. In particular, if $\delta_n : \mathcal{H}_t(M_{n\mathbb{C}}) \to \mathcal{H}_t(M_{m\mathbb{C}})$ and $\gamma_n : L^2(M_n) \to L^2(M_\infty)$ are the canonical embedding then $\delta_n \circ H_{t,n} = H_{t,\infty} \circ \gamma_n$.

7. The Segal-Bargman Transform on the Direct Limit of $\{L^2(M_n)^{K_n}\}_n$

We continue using the notations as in the previous section. We pointed out earlier that the U_n-embedding $V_\mu \hookrightarrow V_{\iota_{n,m}(\mu)}$ does not map $V_\mu^{K_n}$ into $V_{\iota_{n,m}(\mu)}^{K_m}$. This implies that

$$\gamma_{n,m}(L^2(M_n)^{K_n}) \not\subset L^2(M_m)^{K_m}$$

and $\gamma_{n,m}(\psi_\mu) \neq \psi_{\iota_{n,m}(\mu)}$. Therefore, to describe the limit of the heat transform on the level of K-invariant functions, a new embedding is needed. We therefore define $\eta_{n,m} : L^2(M_n)^{K_n} \to L^2(M_m)^{K_m}$ by

$$\eta_{m,n}(f) := \sum_{\mu\in\Lambda^+} d(\iota_{n,m}(\mu))\sqrt{\frac{d(\mu)}{d(\iota_{n,m}(\mu))}}\widehat{f}(\mu)\psi_{\iota_{n,m}(\mu)} .$$

All $\eta_{m,n}$ are linear and it follows from Theorem 2.4 that $\eta_{m,n}$ is an isometric embedding. Furthermore, if $n \leq m \leq p$, then $\eta_{n,p} = \eta_{m,p} \circ \eta_{n,m}$. Therefore $\{L^2(M_n)^{K_n}, \eta_{n,m}\}$ is a direct system of Hilbert spaces. Define (by abuse of notation)

$$L^2(M_\infty)^{K_\infty} := \varinjlim L^2(M_n)^{K_n}$$

in the category of Hilbert spaces and isometric embeddings from the above direct system. Denote by $\eta_n : L^2(M_n) \to L^2(M_\infty)^{K_\infty}$ the resulting canonical embedding.

For $m \geq n$ define

$$\phi_{n,m} : \mathcal{H}_t(M_{n\mathbb{C}})^{K_{n\mathbb{C}}} \longrightarrow \mathcal{H}_t(M_{m\mathbb{C}})^{K_{m\mathbb{C}}}$$

by

$$\sum_{\mu\in\Lambda^+} d(\mu)a(\mu)\widetilde{\psi}_\mu \longmapsto \sum_{\mu\in\Lambda^+} d(\iota_{n,m}(\mu))\sqrt{\frac{d(\mu)}{d(\iota_{n,m}(\mu))}}\frac{e_n(t,\mu)}{e_m(t,\iota_{n,m}(\mu))}\, a(\mu)\widetilde{\psi}_{\iota_{n,m}(\mu)}\,.$$

By Theorem 3.11 it follows that $\phi_{n,m}$ is an isometric U_n-embedding: If

$$F := \sum_{\mu\in\Lambda^+} d(\mu)a(\mu)\widetilde{\psi}_\mu \in \mathcal{H}_t(M_{n\mathbb{C}})^{K_{n\mathbb{C}}}$$

then:

$$\begin{aligned}
\|\phi_{n,m}(F)\|^2_{m,t} &= \sum_{\mu\in\Lambda^+} d(\iota_{n,m}(\mu))\, e_m(2t,\iota_{n,m}(\mu)) \left|\sqrt{\frac{d(\mu)}{d(\iota_{n,m}(\mu))}}\, a(\mu)\frac{e_n(t,\mu)}{e_m(t,\iota_{n,m}(\mu))}\right|^2 \\
&= \sum_{\mu\in\Lambda^+} d(\mu)|a_t(\mu)|^2 e_n(2t,\mu) \\
&= \|F\|^2_{n,t}\,.
\end{aligned}$$

Finally, it is easy to verify that if $n \leq m \leq p$, then $\phi_{n,p} = \phi_{m,p} \circ \phi_{n,m}$.

Thus $\mathcal{H}_t(M_{\infty\mathbb{C}})^{K_{\infty\mathbb{C}}} := \varinjlim\{\mathcal{H}_t(M_{n\mathbb{C}})^{K_{n\mathbb{C}}}$ is well defined in the category of Hilbert spaces and isometric embeddings.

Lemma 7.1. *For $m \geq n$, the following diagram is commutative.*

$$\begin{tikzcd}
L^2(M_n)^{K_n} \arrow[r, "\eta_{n,m}"] \arrow[d, "H_{t,n}"'] & L^2(M_m)^{K_m} \arrow[d, "H_{t,m}"] \\
\mathcal{H}_t(M_n^{\mathbb{C}})^{K_n^{\mathbb{C}}} \arrow[r, "\phi_{n,m}"] & \mathcal{H}_t(M_m^{\mathbb{C}})^{K_m^{\mathbb{C}}}
\end{tikzcd}$$

Proof. Let $f = \sum_{\mu \in \Lambda^+} d(\mu)\widehat{f}(\mu)\psi_\mu \in L^2(M_n)^{K_n}$. Then

$$H_{t,n}(f) = \sum_{\mu} d(\mu) e_n(-t,\mu)\widehat{f}(\mu)\psi_\mu$$

and

$$\begin{aligned}
\phi_{n,m}(H_{t,n}(f)) &= \sum_{\mu\in\Lambda^+} d(\mu)\sqrt{\frac{d(\mu)}{d(\iota_{n,m}(\mu))}}\,\frac{e_n(t,\mu)}{e_m(t,\iota_{n,m}(\mu))}\,\widehat{f}(\mu)e_n(t,\mu)^{-1}\widetilde{\psi}_{\iota_{n,m}(\mu)} \\
&= \sum_{\mu\in\Lambda^+} d(\mu)\sqrt{\frac{d(\mu)}{d(\iota_{n,m}(\mu))}}\,\widehat{f}(\mu)e_m(t,\iota_{n,m}(\mu))^{-1}\widetilde{\psi}_{\iota_{n,m}(\mu)} \\
&= H_{t,m}(\eta_{n,m}(f))\,.
\end{aligned}$$

$\square$

Using the universal mapping property of the direct limit of Hilbert spaces as in Theorem 6.2, we obtain the following:

Theorem 7.2. *There exists a unique unitary isomorphism*

$$S_{t,\infty} : L^2(M_\infty)^{K_\infty} \to \mathcal{H}_t(M_{\infty\mathbb{C}})^{K_{\infty\mathbb{C}}}$$

such that the diagram

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & L^2(M_n)^{K_n} \arrow[r, "\eta_{n+1,n}"] \arrow[d, "H_{t,n}"'] & L^2(M_{n+1})^{K_{n+1}} \arrow[r] \arrow[d, "H_{t,n+1}"] & \cdots \arrow[r] \arrow[d] & L^2(M_\infty)^{K_\infty} \arrow[d, "S_{t,\infty}"'] \\
\cdots \arrow[r] & \mathcal{H}_t(M_n^{\mathbb{C}})^{K_n^{\mathbb{C}}} \arrow[r, "\phi_{n+1,n}"] & \mathcal{H}_t(M_{n+1}^{\mathbb{C}})^{K_{n+1}^{\mathbb{C}}} \arrow[r] & \cdots \arrow[r] & \mathcal{H}_t(M_{\infty\mathbb{C}})^{K_{\infty\mathbb{C}}}
\end{tikzcd}$$

commutes. Furthermore $\eta_\infty \circ S_{t\infty} = \phi_n \circ H_{t,n}$.

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